① Lagrangian correspondences.

A Lag. corr. from (M,ω_M) to (N,ω_N) is a Lag. submfd. of $M^- \times N$.
$$\omega_{M^-\times N} = -pr_M^*\omega_M + pr_N^*\omega_N.$$

Ex. • Graphs of symplectomorphisms $\varphi: M \to N$
$$Gr\,\varphi \underset{Lag}{\subset} M^- \times N.$$

• Products of Lagrangians: $L \underset{Lag}{\subset} M,\ L' \underset{Lag}{\subset} N \Rightarrow L \times L' \underset{Lag}{\subset} M^- \times N.$

• Symplectic reduction: $(M,\omega) \circlearrowleft G$ $\mu: M \to \mathfrak{g}^*$ moment map.
$\mu^{-1}(0)$ coisotropic in M. image of $\mu^{-1}(0) \xhookrightarrow[i\times\pi]{} M^- \times M/\!\!/G$.
$M/\!\!/G = \mu^{-1}(0)/G$.

<u>Composing Lag. correspondences</u>:

$$\begin{tikzcd} M \arrow[r, bend left, "L"] & N \arrow[r, bend left, "K"] & O \end{tikzcd} \overset{?}{\dashrightarrow} \begin{tikzcd} M \arrow[r, bend left, "L\circ K"] & O \end{tikzcd}$$

Geometric composition:

$$L \times K \subset M^- \times N \times N^- \times O.$$
$$\downarrow$$
$$L \times K \cap \Delta_N \qquad \Delta_N = \{(m,n,n,o)\} \subset M^- \times N \times N^- \times O.$$
$$\downarrow$$
$$Pr_{M^-\times O}(L \times K \cap \Delta_N). =: L \circ K. \qquad (m,n,n,o) \longmapsto (m,o).$$

<u>If</u> 1) $L \times K \pitchfork \Delta_N$ transverse

2) $Pr_{M^-\times O}: L\times K \cap \Delta_N \to M^- \times O$ embedded,

<u>Then</u> $L \circ K \underset{Lag.}{\subset} M^- \times O$. Call this an <u>embedded</u> composition.

(• If 1) fails, no way of knowing $L\times K \cap \Delta_N$ is a submfld of $M^- \times N \times N^- \times O$.)

(• If 2) doesn't hold, would only get immersed Lag. instead.) 

(2.) Wehrheim & Woodward's Quilted Floer homology

- Floer hom. for cyclic seq. of Lag. corresp.s

$$\begin{tikzcd} & M_0 \arrow[dr, "L_{01}"] & \\ M_3 \arrow[ur, "L_3"] & & M_1 \arrow[dl, "L_1"] \\ & M_2 \arrow[ul, "L_2"] & \end{tikzcd} = \underline{L} \rightsquigarrow HF(\underline{L}) := HF(\mathcal{L}, \mathcal{L}')$$

Lag. Floer hom in $\mathcal{M} = M_0^- \times M_1 \times M_2^- \times M_3$

$\mathcal{L} \updownarrow L_{01} \times L_{23}$, $\mathcal{L}' \updownarrow L_{12} \times L_{30}$

- How to get simplifications out of the product structure of Lagrangians and $M_0^- \times M_1 \times M_2^- \times M_3$?

<u>Make Special choices</u>: of Hamilt. perturbations & a.c. structures.
– choose them to be of <u>split type</u>:

$\underline{H} = (H_0, H_1, H_2, H_3)$ H_i a Hamiltonian on M_i

$\underline{J} = (J_0, J_1, J_2, J_3)$ J_i an ω_i-compatible a.c.s on M_i

$\underline{H} \rightsquigarrow \mathcal{H}$ Hamiltonian on $M_0^- \times M_1 \times M_2^- \times M_3 \rightarrow X_{\mathcal{H}} \rightarrow \varphi_{\mathcal{H}}$

$\mathcal{H} = (-H_0, H_1, -H_2, H_3)$; $X_{\mathcal{H}}$: Ham. vec. field ; $\varphi_{\mathcal{H}}$: time 1 flow

$\underline{J} \rightsquigarrow \mathcal{J}$ a.c.s on $\mathcal{M}$.

$\mathcal{J} = (-J_0, J_1, -J_2, J_3)$

Perturbed intersections:

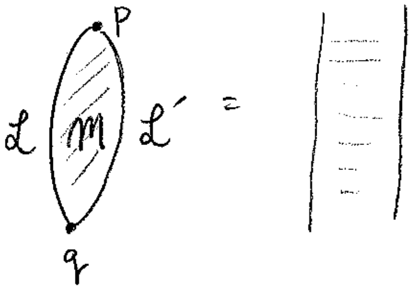
$$I(\mathcal{L}, \mathcal{L}') = \{ p : [0,1] \to \mathcal{M} \mid p(0) \in \mathcal{L},\ p(1) \in \mathcal{L}',\ \dot{p}(t) = X_{\mathcal{H}}(p(t)) \}$$

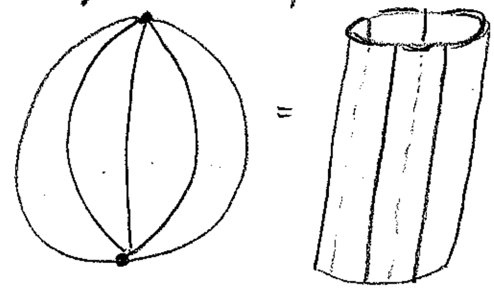
$$\updownarrow$$

$$I(\underline{L}) := \{ \bar{x} = (x_0, x_1, x_2, x_3) \mid x_i : [0,1] \to M_i,\ \dot{x}_i(t) = X_{H_i}(x_i(t)),\ (x_{i-1}(1), x_i(0)) \in L_{i-1,i} \}.$$

- Need to show that it's possible to get $\mathcal{L} \pitchfork \varphi_{\mathcal{H}}(\mathcal{L}')$ using $\mathcal{H}$ of <u>split</u> type.

③ <u>Pseudoholomorphic strips</u> : $\longleftrightarrow$ Pseudoholomorphic quilted strips



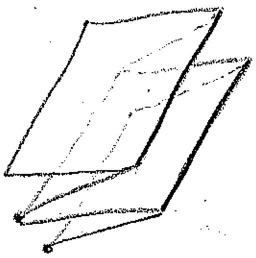


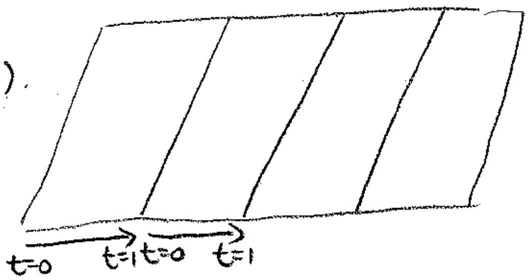
$v : \mathbb{R} \times [0,1] \to \mathcal{M} = M_0^- \times M_1 \times M_2^- \times M_3$
$\quad \shortparallel$
(v_0, v_1, v_2, v_3)

$\underline{u} = (u_0, u_1, u_2, u_3)$
$du_i \circ j = J_i \circ du_i$

$dv \circ j = \mathcal{J} \circ dv \quad \xleftrightarrow{\text{Dictionary}}$

$v_0 = u_0(s, 1-t)$
$v_1 = u_1$
$v_2 = u_2(s, 1-t)$.
$v_3 = u_3$.





$$\partial_s v_0 + (\mathcal{J}_0)\partial_t v_0 = 0 \iff \partial_s u_0 + J_0 \partial_t u_0 = 0.$$

- limits at $s \to \pm\infty$
- $v(s,0) \in \mathcal{L}$
- finite en of v

$\longleftrightarrow$

- limits at $s \to \pm\infty$.
- $(u_i(s,1), u_{i+1}(s,0)) \in L_{i,i+1}$
- finite energy of each u_i

$$HF(\underline{L}) = H\Big(\bigoplus_{\underline{x} \in I(\underline{L})} \boxed{\mathbb{K}}\langle \underline{x} \rangle, \ \partial\Big)$$

counts pseudohol. quilted strips.

* <u>Transversality</u>: Need to show that transversality can be achieved for <u>$\underline{J}$</u> <u>of split type</u>.
? This is not so straightforward (WW – Quilted Floer trajectories w/ constant components).

(Reason is you might have pseudohol. strips w/ const. components,

$|u_0|u_1|u_2|u_3|$ e.g. u_3 = constant, u_0, u_1, u_2 <u>not</u> constant.)

* Everything else (compactness, gluing, etc) ~~is~~ can be based on a local arguments/estimates ~~so~~ so is O.K., with usual caveats about sphere & disk bubbling.
(So, e.g., impose monotonicity/exactness assumptions.)